Lecture 6

Monday, 12 September 2022   11:09 AM

Proof of BFPT

Brouwer's Fixed-Pt. Theorem: Let $X \subseteq \mathbb{R}^n$ be compact, convex, & $f: X \to X$ is continuous. Then f has a fixed pt.

Sperner's Lemma

In 2-D

Consider a triangulation of a triangle in the plane, and a 3-colouring of the vertices of the triangulation that satisfies:

(i) Each vtx. of the outer triangle is a different colour,

(ii) A vertex on an edge of the outer triangle takes the colour of either vertex of the edge.

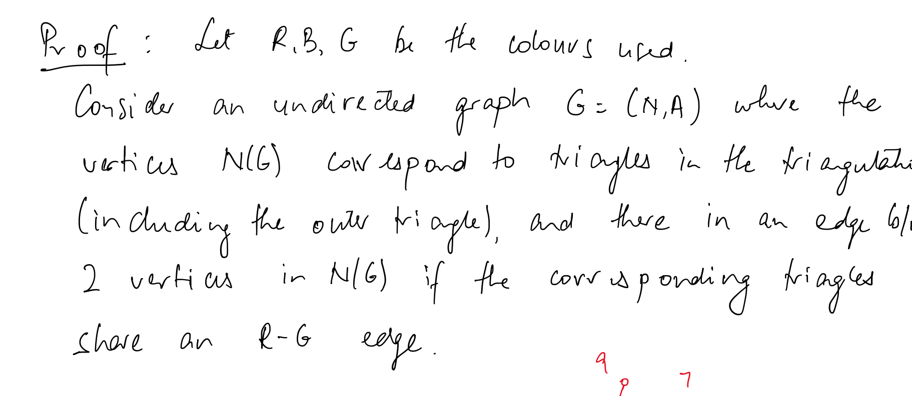
(This is called a Sperner Colouring)

Then the number of trichromatic triangles (including the outer triangle) is even.

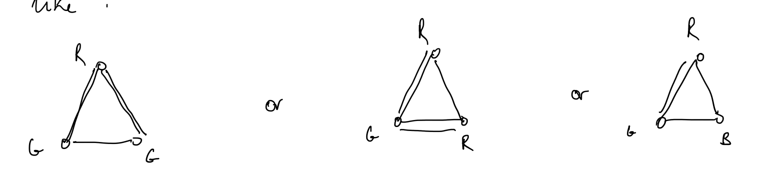
In particular, there is another trichromatic triangle.

Proof: Let R, B, G be the colours used.

Consider an undirected graph $G = (N, A)$ where the vertices $N(G)$ correspond to triangles in the triangulation (including the outer triangle), and there is an edge b/w 2 vertices in $N(G)$ if the corresponding triangles share an R-G edge.



Since each triangle in triangulation has at most 3 neighbours with which it shares an edge, every vertex in $N(G)$ has degree ≤ 3 (except for the outer Δ). Additionally any vertex in $N(G)$ w/ degree ≥ 1 looks like:



i.e., degree of each vtx. is either 0, 1, or 2. A degree 1 vertex is trichromatic

Claim: The vertex corresponding to the outer triangle (say v_0) has odd degree in $N(G)$

Proof: Any vtx. neighbouring v_0 is a triangle on the R-G outer triangle edge (and vice versa). The number of such triangles must be odd (by a simple switching argument)

Now, any graph has an odd # of odd-degree nodes. Removing v_0, we obtain that the triangulation must have an odd # of trichromatic triangles.

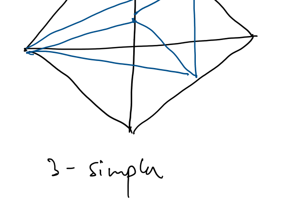
This proves 2-D Sperner's Lemma.

But Sperner's Lemma is true more generally.

n-Simplex: n-dimensional polytope w/ n+1 facets, (faces of dimension n-1)

Eg. triangle is a 2-simplex.

Given an n-simplex, a triangulation divides it into smaller n-simplices



3-simplex

Given an n-simplex, and a division into smaller simplices, let $w_1, \dots, w_{n+1}$ be the outer vertices. Then a Sperner colouring is an $n+1$ colouring $c: V \to [n+1]$ of the vertices of the simplices that satisfies:

(i) $c(w_i) = i$ (each vtx. of the outer simplex is a diff. colour)

(ii) If vertex v is in the convex hull of $\{w_i\}_{i \in S \subseteq [n+1]}$ then $c(v) \in \{c(w_i)\}_{i \in S}$

Sperner's Lemma: The number of panchromatic simplices (w/ each vtx. a diff. colour), including the outer simplex, is even

Hence $\exists$ an inner panchromatic simplex.

Proof is by induction on n (and some geometry)

We will prove that:

Let $f: \Delta_n \to \Delta_n$ be continuous, then f has a fixed pt.

(this is actually the same as BFPT)

Proof: Given Δ_n, $f: \Delta_n \to \Delta_n$, Consider $x \in \Delta_n$.

Then $\exists j \in [n]: x_j > 0$ & $f(x)_j \leq x_j$

If not: $\forall j \in [n]$, either $x_j = 0$, or $f(x)_j > x_j$

but then $\|x\|_1 = 1 < \|f(x)\|_1$.

This is a contradiction.

Then $\forall x \in \Delta_n$, define $\hat{c}(x) = \{j \in [n] : x_j > 0, f(x)_j \leq x_j\}$

Let T be a triangulation of Δ_n, $V(T)$ be vertices of the triangulation, & $e_1, \dots, e_{n+1}$ be the vertices of Δ_n.

Note that $\hat{c}(e_i) = \{i\}$

Then define colouring $c: V(T) \to [n+1]$

as $c(v)$ = some arbitrarily chosen colour from $\hat{c}(v)$

Claim: $c(v)$ is a Sperner colouring.

Proof: Since $c(e_i) = i$, each vertex of Δ_n is a diff colour. Further if $v \in CH(e_{i_1}, \dots, e_{i_t})$, then $i_1, \dots, i_t$ are the only coordinates > 0, hence $c(v) \in \{i_1, \dots, i_t\}$

Then $\exists$ a smaller panchromatic simplex.

Now, we construct a sequence of triangulations $T_1, T_2, \dots$ of Δ_n so that the size of any edge goes to 0.

Each T_i has a panchromatic simplex.

Let $x(j)$ be the centroid of (any) panchromatic simplex of $T(j)$. Consider the sequence $x(1), \dots, x(j), \dots$

This sequence may not be convergent.

Bolzano-Weierstrass' Theorem: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Hence $\exists j_1 < j_2 < j_3 < \dots$ s.t. $x(j_1), x(j_2), \dots$ is convergent

Let's assume for simplicity that $x_1, x_2, \dots$ is convergent, & $\lim_{t \to \infty} x_t = \hat{x}$. Since Δ_n is compact, $\hat{x} \in \Delta_n$

Claim: $f(\hat{x}) = \hat{x}$.

Proof: Suppose not, then $\exists k: f(\hat{x})_k > \hat{x}_k$. Let $\varepsilon = f(\hat{x})_k - \hat{x}_k > 0$. Since f is continuous, $\exists \delta > 0$ s.t. $\forall x: \|x - \hat{x}\| \leq \delta$, $\|f(x) - f(\hat{x})\| \leq \varepsilon/10$ (and hence, $\|f(x)_k - f(\hat{x})_k\| \leq \varepsilon/10$.)

Each $x(i)$ is in a panchromatic simplex.

Let $y(i)$ be the vertex coloured k. Hence $y(i)_k > 0$

$f(y(i))_k \leq y(i)_k$

Since each simplex is getting smaller, $\exists I: \forall i > I$,

$\|y(i) - x(i)\| \leq \varepsilon\delta/10$

Further since $x(i) \to \hat{x}$, $\exists I': \forall i > I'$,

$\|x(i) - \hat{x}\| \leq \varepsilon\delta/10$

Hence $\exists I'': \forall i > I''$, $\|y(i) - \hat{x}\| \leq \varepsilon\delta/5 < \delta$

$\Rightarrow \|f(y(i))_k - f(\hat{x})_k\| \leq \varepsilon/10$

But: since $f(\hat{x})_k \geq \hat{x}_k + \varepsilon$

$\Rightarrow f(y(i))_k \geq y(i)_k + \varepsilon - \frac{\varepsilon\delta}{5} - \frac{\varepsilon}{10} \geq \frac{\varepsilon}{2}$

But we know that $f(y(i))_k \leq y(i)_k$, since it is coloured k. ∎